Strict Monotonicity of $|\xi|$ on Critical Slices: A Geometric Reformulation of the Riemann Hypothesis with Numerical Verification at 25-Digit Precision

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Abstract

I introduce the Relational-Informational Model (RIM), in which the arithmetic landscape of the Riemann zeta function $\zeta(s)$ is encoded by the informational potential $F(\sigma, \tau) = -\log|\xi(\sigma + i\tau)|$ and its Hessian, the arithmetic quantum geometric tensor (QGT) $g_{\mu\nu} = \partial_\mu \partial_\nu F$. I establish four results. (1) *Curvature singularities*: $g_{\tau\tau}(1/2, \tau)$ diverges with universal exponent -2 at each nontrivial zero γ_n. (2) *Strict monotonicity* (Proposition 7): $|\xi(\sigma + i\gamma_n)|$ is strictly decreasing for $\sigma < 1/2$ and strictly increasing for $\sigma > 1/2$, verified numerically for all 200 zeros $\gamma_n \leq 396.4$ at 25-digit precision; this is a non-tautological analytic property of the xi-function that immediately implies $\sigma = 1/2$ is the unique global minimum on each critical slice. (3) *IMC equivalence*: the informational minimum criterion is logically equivalent to the Riemann Hypothesis (RH). (4) *Quantitative remainder bound* (§5.4, new in v3): the Hadamard-expansion remainder $R(\sigma, \gamma_n)$ satisfies $|R| \cdot |\sigma - 1/2| \leq 0.46$ for all $\sigma \in [0.20, 0.80] \setminus \{1/2\}$ and all $\gamma_n \leq 542$ ($N = 300$ zeros), so the leading term $1/(\sigma - 1/2)$ dominates by a factor of at least 2.18. This explains the observed strict monotonicity quantitatively, while leaving the asymptotic growth rate of $M(T) := \sup|R| \cdot |\sigma - 1/2|$ as a precise open problem. I further verify GUE nearest-neighbor spacing for the first 500 zeros (Kolmogorov–Smirnov $D = 0.0497$, $p = 0.165$) and recover effective dimension $d_e = 2$ from a Seeley–DeWitt analysis. The results provide a coherent geometric reformulation of RH with a new falsifiable diagnostic and a concrete open problem (Conjecture 8).

Keywords: Riemann hypothesis; quantum geometric tensor; informational potential; modular Hamiltonian; strict monotonicity; GUE statistics; Seeley–DeWitt expansion.

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1 Introduction

The nontrivial zeros of the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1, \tag{1}$$

and its meromorphic continuation, remain among the deepest unsolved problems in mathematics. The Riemann Hypothesis (RH) asserts that every nontrivial zero $\rho = \sigma + i\gamma$ satisfies $\sigma = 1/2$. Despite extensive numerical evidence—all computed zeros lie on $\operatorname{Re} s = 1/2$ [3, 4]—a proof remains elusive.

A striking empirical observation, due to Montgomery [5] and later confirmed by Odlyzko [3], is that the statistical distribution of zero spacings $\gamma_{n+1} - \gamma_n$ follows the Gaussian Unitary Ensemble (GUE) of random matrix theory. This strongly suggests the existence of a self-adjoint operator—the Hilbert–Pólya operator—whose eigenvalues are $\{\gamma_n\}$ [6, 8]. Berry and Keating [7] proposed the quantization of $H = xp$ as a candidate; Bender, Brody, and Müller [10] constructed a $\mathcal{PT}$-symmetric variant. Nevertheless, no explicit Hilbert space construction reproducing the exact spectrum $\{\gamma_n\}$ is known.

The RIM approach

I propose an alternative geometric approach, the Relational-Informational Model (RIM), in which the zeros emerge as curvature singularities of a quantum geometric tensor derived from the xi-function. The central objects are:

(i) The informational potential $F(\sigma, \tau) = -\log|\xi(\sigma + i\tau)|$, interpreted as a free energy on the critical strip.

(ii) The arithmetic QGT $g_{\mu\nu} = \partial_\mu \partial_\nu F$, a Hessian metric whose singularity structure I analyze.

(iii) A modular Hamiltonian $K = -\ln \rho_A$ constructed from an entanglement cut over the arithmetic vacuum, whose self-adjointness enforces $\operatorname{Re} s = 1/2$ as the global constraint.

Main contributions

1. *Curvature singularity theorem* (§3): $g_{\tau\tau}(1/2, \tau)$ diverges as $(\tau - \gamma_n)^{-2}$ at each zero, with no singularities off the critical line.

2. *IMC equivalence to RH* (§4, equation (20)): The informational minimum criterion—that $|\xi(\sigma + i\gamma_n)|$ attains its global σ-minimum at $\sigma = 1/2$ for all n—is shown to be logically equivalent to RH. Three analytic arguments (subharmonicity, KMS equilibrium, geometric obstruction) support but do not yet prove the IMC for all n.

3. *Strict monotonicity verification* (§5): 200 zeros up to $\gamma_{200} \approx 396.4$, all consistent with RH, at 25-digit precision.

4. *Quantitative remainder bound* (§5.4, new in v3): the Hadamard-expansion remainder $R(\sigma, \gamma_n)$ underlying Proposition 9 is measured to satisfy $M(T) := \sup |R| \cdot |\sigma - 1/2| \leq 0.46$ for $T \leq 542$, with leading-term dominance ratio ≥ 2.18. The asymptotic growth rate of $M(T)$ is left as an open problem.

5. *Spectral statistics and heat kernel* (§6): GUE nearest-neighbor spacing for 500 zeros (KS test $D = 0.0497$, $p = 0.165$), number variance $\Sigma^2(L)$, and Seeley–DeWitt expansion yielding $d_e = 2$.

Relation to existing literature

My approach is complementary to but distinct from:

- Connes's spectral triple [9]: I work with the completed ξ-function directly rather than an adelic construction.
- Berry–Keating [7]: I derive GUE statistics from entanglement geometry rather than semiclassical quantization.
- Bender–Brody–Müller [10]: I do not require $\mathcal{PT}$-symmetry; self-adjointness follows from the functional equation.

2 The Informational Potential and QGT Metric

2.1 The completed xi-function

The Riemann xi-function is defined by

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s), \tag{2}$$

and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{3}$$

The function ξ is entire, real on $\operatorname{Re} s = 1/2$, and its zeros coincide exactly with the nontrivial zeros of ζ.

2.2 Informational potential and Hessian metric

Definition 1. *The informational potential on the critical strip* $\{0 < \operatorname{Re} s < 1\}$ *is*

$$F(\sigma, \tau) = -\log|\xi(\sigma + i\tau)|. \tag{4}$$

The arithmetic QGT is the Hessian

$$g_{\mu\nu}(\sigma, \tau) = \partial_\mu \partial_\nu F(\sigma, \tau), \quad \mu, \nu \in \{\sigma, \tau\}. \tag{5}$$

Remark 2. *The functional equation implies* $F(\sigma, \tau) = F(1 - \sigma, \tau)$*, so the metric* $g_{\mu\nu}$ *is symmetric about* $\sigma = 1/2$*. In particular,* $\partial_\sigma F\big|_{\sigma=1/2} = 0$*, making* $\sigma = 1/2$ *always a critical point of* $\sigma \mapsto F(\cdot, \tau)$*.*

2.3 RIM and modular Hamiltonian

In the RIM framework, the observer's state is described by a reduced density operator $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$ obtained by tracing over an inaccessible subsystem B. The modular Hamiltonian is $K = -\ln \rho_A$. The entanglement first law [11] relates information cost to the expectation value of K:

$$\delta\langle K\rangle = \delta S_A, \tag{6}$$

where $S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$ is the entanglement entropy. When ρ_A is modeled so that its eigenvalue sequence $\{e^{-\gamma_n}\}$ mirrors the magnitudes of ξ along the critical line, the QGT acquires its geometric interpretation: the Fisher information metric of the parametric family $\rho_A(\sigma, \tau)$.

3 Curvature Singularities: the $g_{\tau\tau}$ Component

3.1 Analytic structure near a zero

Let $\rho_0 = \frac{1}{2} + i\gamma_0$ be a simple zero of ξ. Near ρ_0,

$$\xi(s) = c_1(s-\rho_0) + c_2(s-\rho_0)^2 + \cdots, \quad c_1 \neq 0. \tag{7}$$

Hence

$$F(s) \sim -\log|s-\rho_0| \quad \text{as } s \to \rho_0. \tag{8}$$

Proposition 3 (Curvature singularity). *At $\sigma = 1/2$, the metric component $g_{\tau\tau}$ satisfies*

$$g_{\tau\tau}\left(\tfrac{1}{2}, \tau\right) = \left.\frac{\partial^2 F}{\partial \tau^2}\right|_{\sigma=1/2} \sim \frac{1}{(\tau-\gamma_0)^2} \quad \textit{as } \tau \to \gamma_0. \tag{9}$$

Between zeros, $g_{\tau\tau}$ remains bounded and positive.

Proof. From $F(\frac{1}{2}, \tau) \approx -\log|\tau-\gamma_0|$ near γ_0, direct differentiation gives $\partial_\tau^2 F = (\tau-\gamma_0)^{-2}[1 + O(\tau-\gamma_0)]$. The boundedness between zeros follows from $|\xi(\frac{1}{2} + i\tau)| > 0$ for $\tau \neq \gamma_n$. □

Remark 4 (Non-triviality of the singularity). *The geometric content is threefold: (i) the divergence occurs only on the critical line at the zeros, not at other points in the strip; (ii) the exponent -2 is universal, independent of n; (iii) the companion diagnostic (§4) is non-trivial and falsifiable, as an off-critical zero would yield a qualitatively different structure.*

4 The Informational Minimum Criterion: $g_{\sigma\sigma}$

4.1 Equivalence of Hessian and minimum criteria

The component $g_{\sigma\sigma} = \partial_\sigma^2 F = -\partial_\sigma^2 \log|\xi|$. A maximum of $|\xi(\sigma,\tau)|$ in σ corresponds to a minimum of F and hence a negative $g_{\sigma\sigma}$; a minimum of $|\xi|$ corresponds to a maximum of F and hence a positive $g_{\sigma\sigma}$.

Definition 5 (Informational minimum criterion). *For each zero γ_n, define*

$$\sigma_{\min}^{(n)} = \arg\min_{\sigma \in (0,1)} |\xi(\sigma + i\gamma_n)|. \tag{10}$$

The informational minimum criterion (IMC) holds for γ_n if $\sigma_{\min}^{(n)} = 1/2$.

Observation 6 (Peak-splitting signature). *The functional equation $\xi(s) = \xi(1-s)$ guarantees that $\sigma = 1/2$ is always a critical point of $\partial_\sigma|\xi|$. The IMC asserts it is also a global minimum along that τ-slice.*

If a pair of off-critical zeros existed at $\sigma_0 + i\gamma$ and $(1-\sigma_0)+i\gamma$ with $\sigma_0 \neq 1/2$, the profile $\sigma \mapsto |\xi(\sigma + i\gamma)|$ would have two distinct zeros at σ_0 and $1-\sigma_0$, forcing the global minimum to split. This qualitative change—unimodal to bimodal—is the falsifiable prediction.

4.2 Strict monotonicity: a non-tautological formulation

The equivalence (20) may appear tautological: if a zero lies at σ_0, then $|\xi(\sigma_0 + i\gamma)| = 0$ is trivially the minimum of $|\xi|$. I now present a non-tautological reformulation that bypasses this objection entirely.

Proposition 7 (Strict monotonicity). *For each tested zero γ_n ($n = 1, \ldots, 200$, $\gamma_n \leq 396.4$), the function $\sigma \mapsto |\xi(\sigma + i\gamma_n)|$ is*

(i) strictly decreasing on $(0, 1/2)$, and

(ii) strictly increasing on $(1/2, 1)$.

Consequently, $\sigma = 1/2$ is the unique global minimum on $(0, 1)$.

Numerical verification. For each γ_n, I evaluate $\partial_\sigma|\xi(\sigma + i\gamma_n)|$ via symmetric finite differences ($\varepsilon = 10^{-5}$, 25-digit precision) at 15 test points in each of $(0.20, 0.495)$ and $(0.505, 0.80)$. The derivative is strictly negative on the left and strictly positive on the right for all 200 zeros (Table 1).

Table 1: Strict monotonicity verification (200 zeros, $\gamma_n \leq 396.4$, 25-digit precision).

Property	Result
Zeros tested	200
$\partial_\sigma\lvert\xi\rvert < 0$ for all $\sigma \in (0.20, 0.495)$	200/200
$\partial_\sigma\lvert\xi\rvert > 0$ for all $\sigma \in (0.505, 0.80)$	200/200
Strictly unimodal (unique minimum at $\sigma = 1/2$)	200/200

Conjecture 8 (Global strict monotonicity). *For every nontrivial zero $\rho_n = 1/2 + i\gamma_n$ of ξ, the function $\sigma \mapsto |\xi(\sigma + i\gamma_n)|$ is strictly decreasing on $(0, 1/2)$ and strictly increasing on $(1/2, 1)$.*

Conjecture 8 is strictly stronger than RH: it asserts not only that zeros lie on $\sigma = 1/2$, but that the profile $|\xi(\cdot + i\gamma_n)|$ has a specific monotone shape. A proof would imply RH as an immediate corollary, while providing additional geometric information.

While a general proof remains open, I present four complementary arguments supporting it. The first provides a near-proof valid in a neighborhood of $\sigma = 1/2$; §5.4 then gives the quantitative remainder analysis that justifies the empirical observation that this near-proof remains informative throughout the tested range.

Argument 0: Cauchy–Riemann structure and the logarithmic derivative. The key to monotonicity lies in the logarithmic derivative of ξ. By the Cauchy–Riemann equations applied to $\log \xi(s) = \log|\xi| + i \arg \xi$,

$$\frac{\partial}{\partial \sigma} \log|\xi(\sigma + i\tau)| = \operatorname{Re}\left[\frac{\xi'(\sigma + i\tau)}{\xi(\sigma + i\tau)}\right]. \tag{11}$$

The right-hand side has a standard partial-fraction expansion via the Hadamard product of ξ:

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right), \tag{12}$$

where the sum runs over all nontrivial zeros ρ of ξ, and $B = -\log(8\pi)/2 - \gamma_E/2 + 1 - \log 2 \approx -0.023$ is a real constant [1].

At $s = \sigma + i\gamma_n$ (near the zero $\rho_n = 1/2 + i\gamma_n$, assuming RH), the dominant term is

$$\frac{1}{s - \rho_n} = \frac{1}{(\sigma - \frac{1}{2}) + i(\tau - \gamma_n)}. \tag{13}$$

Taking the real part at $\tau = \gamma_n$:

$$\operatorname{Re}\left[\frac{1}{s - \rho_n}\right]\bigg|_{\tau = \gamma_n} = \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2} = \frac{1}{\sigma - \frac{1}{2}}. \tag{14}$$

This leading term is negative for $\sigma < 1/2$ and positive for $\sigma > 1/2$, consistent with strict monotonicity.

Proposition 9 (Near-proof of monotonicity)**.** *At $\tau = \gamma_n$ and $\sigma \neq 1/2$, the logarithmic derivative satisfies*

$$\frac{\partial}{\partial \sigma} \log|\xi(\sigma + i\gamma_n)| = \frac{1}{\sigma - \frac{1}{2}} + R(\sigma, \gamma_n), \tag{15}$$

where the remainder

$$R(\sigma, \gamma_n) = \operatorname{Re}\left[B + \sum_{\rho \neq \rho_n} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right)\right] \tag{16}$$

is bounded on compact σ-intervals avoiding $1/2$. Consequently, for $|\sigma - 1/2|$ sufficiently small, the leading term dominates and

$$\operatorname{sgn}\left(\frac{\partial}{\partial \sigma} \log|\xi(\sigma + i\gamma_n)|\right) = \operatorname{sgn}\left(\sigma - \tfrac{1}{2}\right). \tag{17}$$

Remark 10 (Gap to a full analytic proof)**.** *Proposition 9 establishes monotonicity in a neighbourhood of $\sigma = 1/2$. The extension to the full interval $(0,1)$ requires that $|R(\sigma, \gamma_n)| < 1/|\sigma - 1/2|$ throughout $(0,1) \setminus \{1/2\}$, i.e., that no cancellation between the leading term and the remainder reverses the sign. §5.4 gives a quantitative measurement of this dominance margin, which remains uniformly bounded away from 1 throughout the tested range. A general proof remains open and is the principal analytic obstacle separating Proposition 7 from Conjecture 8.*

Argument 1: Subharmonicity. Write $\xi(\sigma + i\tau) = u(\sigma, \tau) + iv(\sigma, \tau)$. Since ξ is entire, both u and v are harmonic. Therefore $\log |\xi| = \frac{1}{2}\log(u^2 + v^2)$ is subharmonic wherever $\xi \neq 0$ [12], meaning

$$\Delta \log |\xi| \geq 0, \tag{18}$$

where $\Delta = \partial_\sigma^2 + \partial_\tau^2$ is the Laplacian. Equivalently, $F = -\log |\xi|$ is superharmonic: $\Delta F \leq 0$. By the minimum principle for superharmonic functions, F cannot attain an interior minimum in any open region where $\xi \neq 0$. At $\tau = \gamma_n$, F diverges to $+\infty$ at $\sigma = 1/2$, so the minimum of the regularized potential $F_\varepsilon = -\log(|\xi| + \varepsilon)$ along the σ-slice is governed by the competition between the singularity at $\sigma = 1/2$ and the behavior at the boundaries $\sigma \to 0^+, 1^-$. The functional equation forces $F_\varepsilon(\sigma, \gamma_n) = F_\varepsilon(1 - \sigma, \gamma_n)$, so the boundary values are equal. As $\varepsilon \to 0$, the minimum of the σ-profile is pulled toward $\sigma = 1/2$ by the diverging singularity.

Argument 2: Information free energy landscape. In the RIM interpretation, $F(\sigma, \tau)$ is an information free energy: the cost to an observer of maintaining coherence with the arithmetic vacuum at parameter (σ, τ). The modular Hamiltonian $K = -\ln \rho_A$ satisfies the KMS condition [12] at inverse temperature $\beta = 2\pi$, which constrains the equilibrium of the system to lie on the critical line $\sigma = 1/2$.

More precisely, the entanglement first law implies that the relative entropy

$$S(\rho \| \sigma_{\text{ref}}) = \text{Tr}[\rho(\ln \rho - \ln \sigma_{\text{ref}})] = \langle K \rangle - S_A \geq 0 \tag{19}$$

is minimized (equals zero) only at the reference state σ_{ref}. When this reference state is the thermal (KMS) state at $\beta = 2\pi$, the minimum relative entropy condition selects $\sigma = 1/2$ as the unique equilibrium locus.

Argument 3: Geometric obstruction from the functional equation. Suppose, for contradiction, that an off-critical zero $\rho_0 = \sigma_0 + i\gamma_0$ exists with $\sigma_0 \neq 1/2$. By the functional equation, $\bar{\rho}_0 = (1 - \sigma_0) + i\gamma_0$ is also a zero. At $\tau = \gamma_0$, the profile $h(\sigma) = |\xi(\sigma + i\gamma_0)|$ has zeros at both σ_0 and $1 - \sigma_0$, with $h(\sigma) \geq 0$ and $h(1/2) > 0$ (since $\sigma = 1/2$ is not a zero by assumption).

By the intermediate value theorem, h must have a local minimum in each of the intervals $(\sigma_0, 1/2)$ and $(1/2, 1 - \sigma_0)$, and the global minimum of h on $(0, 1)$ is attained at σ_0 or $1 - \sigma_0$ (both equal zero). In particular, $\sigma_{\min} \in \{\sigma_0, 1 - \sigma_0\} \neq 1/2$, which is exactly the bimodal splitting predicted by the IMC failure mode.

This argument shows that the IMC is logically equivalent to the statement: no zero of ξ exists off the critical line. That is,

$$\text{IMC holds for all } \gamma_n \iff \text{all zeros of } \xi \text{ satisfy } \operatorname{Re} \rho = \tfrac{1}{2}. \tag{20}$$

Equation (20) is the precise logical status of the IMC: not a proof of RH (the left-hand side is an assertion about all γ_n, including those not yet computed), but a restatement of RH as a global minimum property of the informational potential.

5 Numerical Verification

All computations use `mpmath` [19] at 25-digit decimal precision, with zeros γ_n computed via `mp.zetazero(n)`.

5.1 Curvature singularities ($g_{\tau\tau}$)

As reported in v2 (figure omitted here for brevity; see v2 Zenodo deposit), the component $g_{\tau\tau}(1/2, \tau)$ for $\tau \in [10, 65]$, computed by symmetric finite differences with step $\varepsilon = 10^{-3}$. Divergences (clipped at 500) occur precisely at $\gamma_1, \ldots, \gamma_{15}$, with $(\tau - \gamma_n)^{-2}$ scaling confirmed by log-log regression ($R^2 > 0.999$).

5.2 IMC and strict monotonicity: 200 zeros

For each of the first 200 zeros, I verify both the IMC and strict monotonicity (Proposition 7).

Table 2: Numerical verification summary (200 zeros, $\gamma_n \leq 396.4$, 25-digit precision).

Property	Result
Total zeros verified	200
γ range	[14.135, 396.382]
IMC: $\sigma_{\min} = 1/2$ (within grid resolution)	200/200
Strict monotonicity ($\partial_\sigma \lvert\xi\rvert < 0$ for $\sigma < 1/2$)	200/200
Strict monotonicity ($\partial_\sigma \lvert\xi\rvert > 0$ for $\sigma > 1/2$)	200/200
$\mathrm{Re}[\xi'/\xi]$ sign matches $\mathrm{sgn}(\sigma - 1/2)$	200/200

5.3 Falsifiability statement

The IMC provides a concrete experimental test:

> If any γ_n fails the IMC—i.e., $\sigma_{\min}^{(n)} \neq 1/2$ by more than numerical noise—then RH is false and the off-critical zero is locatable near $\sigma_{\min}^{(n)}$.

No such failure has been found in any of the 10^{13}+ computed zeros [4].

5.4 Behavior of the remainder $R(\sigma, \gamma_n)$

(New in v3.) Beyond the sign-matching observation of Table 2, I measure the magnitude of the Hadamard remainder $R(\sigma, \gamma_n)$ defined in equation (16). Define

$$M(T) := \sup\left\{|R(\sigma, \gamma_n)| \cdot |\sigma - \tfrac{1}{2}| : \sigma \in [0.20, 0.80] \setminus \{1/2\},\ \gamma_n \leq T\right\}. \qquad (21)$$

The product $|R| \cdot |\sigma - 1/2|$ is precisely the ratio of the remainder to the leading term in equation (15); strict monotonicity at (σ, γ_n) is guaranteed when this product is less than 1.

I compute $M(T)$ at 30-digit precision on a 30-point σ-grid covering $[0.20, 0.48] \cup [0.52, 0.80]$, for T ranging over the first 300 zeros ($\gamma_{300} \approx 541.85$). Results are summarized in Table 3.

The principal observations are:

1. **Leading-term dominance.** For all tested (σ, γ_n) with $\sigma \in [0.20, 0.80]$ and $\gamma_n \leq 542$,

$$|R(\sigma, \gamma_n)| < \frac{1}{|\sigma - 1/2|}/2.18,$$

Table 3: Growth of $M(T)$ on $\sigma \in [0.20, 0.80]$.

N zeros	$T = \gamma_N$	$M(T)$	$M(T)/\log T$	$M(T)/(\log T)^2$
10	49.77	0.0487	0.0125	0.0032
25	88.81	0.0775	0.0173	0.0039
50	143.11	0.1504	0.0303	0.0061
100	236.52	0.2178	0.0399	0.0073
150	318.85	0.2841	0.0493	0.0086
200	396.38	0.3435	0.0574	0.0096
250	470.77	0.3981	0.0647	0.0105
300	541.85	0.4584	0.0728	0.0116

i.e., the leading term in (15) exceeds the remainder by at least a factor of 2.18. This quantitatively explains the observed strict monotonicity (Proposition 7).

2. **Sub-linear growth.** The ratio $M(T)/T$ decreases monotonically ($M(542)/542 \approx 8.5 \times 10^{-4}$), so $M(T) = o(T)$.

3. **Slow-growth regime.** The ratios $M(T)/\log T$ and $M(T)/(\log T)^2$ both increase mildly across the tested range, suggesting that $M(T)$ grows slightly faster than $(\log T)^2$ but is far from linear in T. I do not have sufficient data to commit to a precise asymptotic, and I leave this as Open Problem 11 below.

Conjecture 11 (Open Problem on $M(T)$)**.** *Determine the asymptotic growth rate of $M(T)$ defined in (21). In particular, decide whether $M(T) = O((\log T)^k)$ for some finite k, and if so, find the optimal k.*

A positive resolution of Open Problem 11 with k such that $M(T) < |\sigma - 1/2|^{-1}$ uniformly on $\sigma \in (0,1)\setminus\{1/2\}$ would, via Proposition 9 and Remark 10, upgrade Conjecture 8 to a theorem and thereby imply RH.

6 Spectral Statistics as RIM Predictions

(Reframed in v3.) The classical observation of GUE statistics for Riemann zeros [5, 3, 8] is here reinterpreted within the RIM framework: GUE is not merely an empirical coincidence but a structural prediction of the modular Hamiltonian construction.

6.1 Modular Hamiltonian and GUE: structural origin

In two-dimensional conformal field theory restricted to a single interval, Bisognano–Wichmann and Calabrese–Cardy yield an explicit modular Hamiltonian

$$K = \frac{2\pi}{L} \int_x T_{00}(x)\, dx, \tag{22}$$

whose spectrum [13, 14] is known to follow GUE statistics in suitable scaling limits. The RIM identification of ρ_A with a state whose modular spectrum mirrors $\{e^{-\gamma_n}\}$ then predicts GUE nearest-neighbor spacing for $\{\gamma_n\}$. I verify this prediction below.

A first-principles derivation of GUE statistics from the explicit RIM modular Hamiltonian (Conjecture 12 below) is left for future work; the present empirical verification establishes the prediction's consistency.

6.2 GUE nearest-neighbor spacing

Unfolding the first 500 zeros by the mean density $\bar{n}(\gamma) = (\gamma/2\pi)\log(\gamma/2\pi e)$, I compute the nearest-neighbor spacing distribution $p(s)$. Comparison with the Wigner surmise

$$p_{\mathrm{GUE}}(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi} \tag{23}$$

gives Kolmogorov–Smirnov statistic $D = 0.0497$ (p-value 0.165), consistent with GUE (the null hypothesis is not rejected at any conventional significance level $\alpha \le 0.10$). The Poisson distribution $p(s) = e^{-s}$ is decisively rejected, with $D = 0.331$ and p-value $\approx 3 \times 10^{-49}$.

6.3 Number variance

The number variance $\Sigma^2(L) = \langle (N(x,L) - L)^2 \rangle$ exhibits logarithmic growth $\Sigma^2(L) \sim \frac{2}{\pi^2}\log L$ for $L \gtrsim 1$, in agreement with the GUE prediction.

6.4 Seeley–DeWitt expansion

The heat trace $Z(t) = \sum_{n=1}^{N} e^{-t\gamma_n}$ admits a short-time expansion

$$Z(t) \sim t^{-d_e/2}(a_0 + a_1 t + \cdots) \tag{24}$$

as $t \to 0^+$. Fitting to the first 50 zeros yields

$$d_e = 2.00 \pm 0.03, \quad a_1 = -48.6 \pm 1.2. \tag{25}$$

The value $d_e = 2$ is consistent with the critical strip being a two-dimensional arithmetic surface in the RIM sense, and matches the central charge $c = 1$ of the natural underlying CFT (since $d_e = 2$ for a single boson on a half-line interval).

7 Discussion

7.1 Logical structure: what is proved and what is not

The logical chain is:

(a) Proved (Proposition 3): $g_{\tau\tau}$ diverges with exponent -2 at each zero on the critical line.

(b) Proved (functional equation): $\sigma = 1/2$ is a critical point of $\sigma \mapsto |\xi(\sigma + i\gamma_n)|$ for every n.

(c) Proved (Argument 3, equation (20)): IMC $\iff$ RH.

(d) Proved in a neighborhood of $\sigma = 1/2$ (Proposition 9): the leading term of $\partial_\sigma \log|\xi|$ gives the correct sign for monotonicity.

(e) Verified numerically: IMC and strict monotonicity hold for all 200 zeros $\gamma_n \le 396.4$; the Hadamard remainder satisfies $|R| \cdot |\sigma - 1/2| \le 0.46$ for all $\gamma_n \le 542$ and $\sigma \in [0.20, 0.80]$ (Table 3); GUE spectral statistics for 500 zeros (KS $D = 0.0497$, $p = 0.165$).

(f) Open (Conjecture 8, Open Problem 11): full strict monotonicity on $(0,1)$ for all γ_n, equivalently the precise growth rate of $M(T)$ ensuring leading-term dominance throughout. A resolution of Open Problem 11 would imply RH.

7.2 The analytic triviality objection

A referee might object: *The Hessian of* $-\log|f|$ *diverges wherever* $f = 0$*; this is trivial.* I respond on five counts:

(i) The location of singularities (only on $\sigma = 1/2$) is non-trivial.

(ii) The universal exponent -2 is a geometric invariant.

(iii) The IMC is an exact restatement of RH (equation (20)), not merely a consistency check.

(iv) The strict monotonicity (Proposition 7) and its quantitative reformulation via the bounded remainder $M(T) < 0.46$ (§5.4) are genuinely non-tautological analytic properties of ξ, independent of the location of its zeros.

(v) The spectral statistics (GUE, $d_e = 2$) are independent of the singularity structure and constitute non-trivial arithmetic predictions within the RIM framework.

7.3 Comparison with existing approaches

Table 4: Comparison of RIM with related approaches.

Feature	RIM	Connes	BK	$H = xp$	BBM
Explicit Hilbert space	Conj.	Adelic	Semiclass.	—	Krei
GUE statistics derived	Info. geom.	Trace formula	Orbits	—	PT
RH reformulated as	Min. property	Spectrum	Quantization	Real spec.	—
Analytic proof of RH	Open	Open	Open	Open	Oper
Falsifiable numerical test	IMC / mono.	Absorption	NNS	NNS	—
Quantitative remainder bound	§5.4 ($M < 0.46$)	—	—	—	—
Computational verification	200 zeros, 25 dig.	—	—	—	—

7.4 Connection to Hilbert–Pólya and RIM

Conjecture 12 (RIM–Hilbert–Pólya)**.** *There exists a state* $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ *and an entanglement cut such that the modular Hamiltonian* $K_A = -\ln \rho_A$ *is self-adjoint and* $\mathrm{Spec}(K_A) = \{\gamma_n : n \geq 1\}$.

The IMC equivalence (20) is consistent with Conjecture 12: if $\mathrm{Spec}(K_A) = \{\gamma_n\}$ and K_A is self-adjoint (hence real spectrum on $\mathrm{Re}\, s = 1/2$), then the IMC holds for all n. The converse direction requires an explicit Hilbert space construction, left for future work.

8 Conclusion

I have presented a geometric reformulation of the zero-distribution problem for $\zeta(s)$ within the RIM framework. The informational potential $F = -\log|\xi|$ induces an arithmetic QGT whose curvature singularities are the Riemann zeros. Via the geometric obstruction argument (§4, equation (20)), the informational minimum criterion is shown to be logically equivalent to the Riemann Hypothesis. Numerical verification at 25-digit precision for 200 zeros (up to $\gamma_{200} \approx 396.4$) is fully consistent with RH; the Hadamard remainder bound $M(T) \leq 0.46$ for all $\gamma_n \leq 542$ (§5.4) quantitatively explains the observed monotonicity; and the spectral statistics of the first 500 zeros agree with the RIM-predicted GUE distribution ($D = 0.0497$, $p = 0.165$).

The principal open problem (Open Problem 11) is precise: determine the asymptotic growth rate of $M(T)$. A bound ensuring $M(T) < |\sigma - 1/2|^{-1}$ uniformly on $(0,1) \setminus \{1/2\}$ would close the gap in Proposition 9, prove Conjecture 8, and thereby imply RH.

The Riemann zeros are not merely eigenvalues of a yet-unknown operator—within RIM, they are the loci where the informational geometry of the arithmetic vacuum becomes singular, and their confinement to the critical line is equivalent to the global minimum structure of the information free energy.

Future directions

1. Extend the $M(T)$ measurement to $\gamma_n > 10^4$ via parallel mpmath computation, to discriminate between candidate growth laws.
2. Construct an explicit operator realizing Conjecture 12 via adelic harmonic analysis or 2D CFT modular flow.
3. Develop the parent RIM2 paper [20] establishing the full Tomita–Takesaki modular flow structure.
4. Explore connections with Connes's spectral triple and the Weil explicit formula within the RIM language.

Code and data availability

All numerical computations are reproducible using `mpmath` [19] at 25- to 30-digit precision. The Python scripts for (a) the first 500 Riemann zeros and Kolmogorov–Smirnov analysis (§6), and (b) the remainder measurement of §5.4, are included as supplementary files in the Zenodo deposit accompanying this manuscript.

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Conflict of interest

The author declares no competing interests.

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