Anchored Accumulation Calculus: Mellin Diagonalization and Operator-Theoretic Classification of Scale-Covariant Memory

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April 14, 2026

Abstract

We formalize Anchored Accumulation Calculus (AAC), an operator-theoretic framework for scale-covariant memory on the multiplicative group $\mathbb{R}_+$. By defining memory as causal accumulation along exponential scale trajectories, we establish that AAC operators on a dense core are unitarily equivalent to translation-invariant Fourier multipliers. This construction allows us to define the maximal closed extension of these operators, avoiding ad hoc integration-by-parts arguments. We subsequently classify their generators: completely monotone memory kernels are rigorously shown to generate scale-invariant Lévy subordinators via Bernstein's theorem, while heavy-tailed kernels are unitarily equivalent to Mellin-based Hadamard fractional derivatives. Finally, to demonstrate that AAC is not merely a reformulation of Markovian semigroups, we analyze oscillatory memory kernels, proving they generate dispersive, non-dissipative unitary groups on the scale domain.

1 Introduction

Memory effects in systems governed by scale invariance—such as anomalous diffusion on fractal geometries, polymer chain dynamics, and viscoelastic log-price models—are naturally modeled on the multiplicative group $\mathbb{R}_+ = (0, \infty)$. While the Mellin transform is the standard analytical tool for such geometries [3, 6], the operator-theoretic treatment of scale-covariant memory (often framed heuristically via Hadamard fractional calculus [4]) frequently lacks rigorous grounding in domain closure, sectoriality, and complete semigroup generation.

In this paper, we construct *Anchored Accumulation Calculus* (AAC), a rigorous functional-analytic framework that parameterizes memory as a causal Volterra-type accumulation along a backward logarithmic scale trajectory.

To resolve the functional-analytic ambiguities present in informal fractional calculi, we provide three main contributions:

1. **Graph Closure via Unitary Equivalence:** We define the AAC operator on a dense core and formally identify its maximal closed domain via continuous Fourier multipliers.

2. **Lévy–Khintchine Derivation:** Utilizing Bernstein function theory, we rigorously prove that completely monotone kernels generate conditionally negative definite symbols and explicit scale-invariant Lévy measures.

3. **Spectral Classification Beyond Semigroups:** We prove that heavy-tailed kernels correspond to sectorial normal operators (recovering Hadamard calculus via Balakrishnan's spectral theorem), while oscillatory kernels escape the Lévy framework entirely to generate dispersive scale-dynamics.

2 Functional Setting and the Dense Core

Let $H = L^2(\mathbb{R}_+, dx/x)$ denote the Hilbert space equipped with the invariant Haar measure.

Definition 2.1 (Core Domain and Unitary Map)**.** *Let $\mathcal{D}_0 = C_c^\infty(\mathbb{R}_+)$ be the space of smooth functions with compact support in $(0,\infty)$, which is dense in H. We define the logarithmic unitary map $U : H \to L^2(\mathbb{R}, dy)$ by $(Uf)(y) = f(e^y)$. U is an isometric isomorphism mapping $\mathcal{D}_0$ bijectively onto $C_c^\infty(\mathbb{R})$.*

Definition 2.2 (Mellin Transform)**.** *The Mellin transform $M : H \to L^2(\mathbb{R}, d\xi)$ is defined via $M = \mathcal{F} \circ U$, where $\mathcal{F}$ is the standard unitary Fourier transform on $L^2(\mathbb{R})$.*

Assumption 2.1 (Kernel Regularity)**.** *The memory kernel ϕ is a tempered distribution in $\mathcal{S}'(\mathbb{R})$ with $\operatorname{supp}(\phi) \subseteq [0,\infty)$. We assume its distributional Fourier–Laplace transform $\widehat{\phi}(\xi)$ exists such that the multiplier function $m(\xi) = i\xi\widehat{\phi}(\xi)$ is measurable and belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R})$.*

3 The AAC Operator and its Maximal Extension

We define the AAC operator initially on the core domain, where boundary conditions trivially vanish.

Definition 3.1 (AAC Operator on the Core)**.** *For $f \in \mathcal{D}_0$, the AAC operator $\mathcal{T}_\phi^0$ is defined via the causal exponential scale trajectory $\gamma_x(s) = xe^{-s}$:*

$$\mathcal{T}_\phi^0 f(x) = -\int_0^\infty \phi(s)\, \frac{\partial}{\partial s}\left[f(xe^{-s})\right] ds.$$

Because $s \mapsto f(xe^{-s})$ is smooth with compact support in $[0,\infty)$, the action of the distribution ϕ is strictly well-defined.

Lemma 3.1 (Distributional Convolution)**.** *On the core $\mathcal{D}_0$, $U\mathcal{T}_\phi^0 U^{-1}$ acts as the additive convolution operator $g \mapsto \phi * g'$ in $\mathcal{S}'(\mathbb{R})$.*

Proof. Let $g = Uf \in C_c^\infty(\mathbb{R})$. By the chain rule, $-\frac{\partial}{\partial s}[f(xe^{-s})] = -g'(\ln x - s)$. The integral against $\phi(s)$ evaluates the distribution ϕ on the test function $s \mapsto -g'(\ln x - s)$, which is exactly the definition of the convolution $(\phi * g')(\ln x)$. Applying U maps $\ln x \mapsto y$. □

Theorem 3.1 (Maximal Closed Extension)**.** *Under Assumption 2.1, $\mathcal{T}_\phi^0$ is closable in H. Its unique closed extension, denoted $\mathcal{T}_\phi$, has the maximal domain:*

$$D(\mathcal{T}_\phi) = \left\{ f \in H : m(\xi)(Mf)(\xi) \in L^2(\mathbb{R}, d\xi) \right\},$$

equipped with the graph norm $\|f\|^2_{\mathcal{T}_\phi} = \|f\|^2_H + \|m \cdot Mf\|^2_{L^2(\mathbb{R})}$. Furthermore, $\mathcal{T}_\phi$ is unitarily equivalent to the densely defined Fourier multiplier $T_m = \mathcal{F}^{-1}M_m\mathcal{F}$.

Proof. By Lemma 3.1, $U\mathcal{T}_\phi^0 f = \phi * (Uf)'$. Taking the Fourier transform over $C_c^\infty(\mathbb{R})$, we find $\mathcal{F}(\phi * g')(\xi) = \widehat{\phi}(\xi)(i\xi\widehat{g}(\xi)) = m(\xi)\widehat{g}(\xi)$. Thus, on $\mathcal{D}_0$, $\mathcal{T}_\phi^0 = U^{-1}\mathcal{F}^{-1}M_m\mathcal{F}U$. Because $m(\xi)$ is measurable, the maximal multiplier M_m is a closed, densely defined operator on $L^2(\mathbb{R}, d\xi)$. Since U and $\mathcal{F}$ are unitary, the pull-back operator $\mathcal{T}_\phi$ is closed and densely defined on H. The domain $D(\mathcal{T}_\phi)$ follows directly from the domain of M_m. □

4 Lévy–Khintchine Representation

We consider the Cauchy problem $\partial_t f = -\mathcal{T}_\phi f$. For this to generate a C_0-contraction semigroup, the multiplier $m(\xi)$ must be conditionally negative definite (CND) with $\mathrm{Re}(m(\xi)) \geq 0$.

Theorem 4.1 (Lévy–Khintchine Derivation). *Assume $\phi(s)$ is completely monotone, admitting the Bernstein representation $\phi(s) = \int_0^\infty e^{-s\tau}\nu(d\tau)$ where $\int_0^\infty (1 \wedge \tau^{-1})\nu(d\tau) < \infty$. Then $m(\xi)$ is CND, and there exists a unique positive Lévy measure μ such that:*

$$m(\xi) = \int_0^\infty \left(1 - e^{-i\xi y}\right) \mu(dy).$$

Consequently, $-\mathcal{T}_\phi$ generates a scale-invariant Lévy subordinator on H.

Proof. Substituting the Bernstein representation into the symbol $m(\xi) = i\xi \int_0^\infty e^{-i\xi s}\phi(s)ds$ yields:

$$m(\xi) = \int_0^\infty \frac{i\xi}{\tau + i\xi}\nu(d\tau).$$

Using the integral identity $\frac{i\xi}{\tau+i\xi} = \int_0^\infty (1 - e^{-i\xi y})\tau e^{-\tau y}dy$, we apply Fubini–Tonelli to the double integral. The modulus of the integrand is bounded by $|1 - e^{-i\xi y}|\tau e^{-\tau y} \leq (2 \wedge |\xi| y)\tau e^{-\tau y}$. Integrating with respect to y gives a bound proportional to $\tau \int_0^\infty y e^{-\tau y}dy = \tau^{-1}$ for small τ, and 2 for large τ. Because ν integrates $1 \wedge \tau^{-1}$, the Fubini swap is rigorously justified, allowing us to define the measure:

$$\mu(dy) = \left(\int_0^\infty \tau e^{-\tau y}\nu(d\tau)\right) dy.$$

By construction, $\int_0^\infty (1 \wedge y)\mu(dy) < \infty$, satisfying the strict criteria for a Lévy measure [1]. Because $\mathrm{Re}(1 - e^{-i\xi y}) = 1 - \cos(\xi y) \geq 0$, Lumer–Phillips theorem guarantees $-\mathcal{T}_\phi$ generates a C_0-contraction semigroup. □

5 Fractional Powers and Sectoriality

Theorem 5.1 (Sectoriality and Normal Operators). *Let $\phi(s) = \frac{1}{\Gamma(1-\beta)}s^{-\beta}$ for $\beta \in (0, 1)$. The resulting operator $\mathcal{T}_\phi$ is a closed normal operator and is sectorial of angle $\omega = \beta\pi/2$.*

Proof. The distributional transform of $\phi(s)$ yields the multiplier $m(\xi) = (i\xi)^\beta$. To ensure $m(\xi)$ is single-valued and measurable, we define it via the principal branch: $m(\xi) = |\xi|^\beta e^{i\,\mathrm{sgn}(\xi)\beta\pi/2}$ for $\xi \in \mathbb{R} \setminus \{0\}$. Because $m(\xi)$ is measurable, the maximal multiplier T_m is a normal operator [2], and thus $\mathcal{T}_\phi$ is normal on H. Its spectrum $\sigma(\mathcal{T}_\phi)$ is the essential range of $m(\xi)$, specifically the rays $\Gamma = \{re^{\pm i\beta\pi/2} : r \geq 0\}$. This spectrum is entirely contained in the sector $\Sigma_\omega = \{z \in \mathbb{C} : |\arg z| \leq \beta\pi/2\}$. By the spectral theorem for normal operators, the resolvent satisfies $\|(\lambda I - \mathcal{T}_\phi)^{-1}\| \leq \mathrm{dist}(\lambda, \sigma(\mathcal{T}_\phi))^{-1}$ for $\lambda \notin \Sigma_\omega$, satisfying the definition of a sectorial operator. □

Corollary 5.1 (Mellin-Hadamard Equivalence). *Because $\mathcal{T}_\phi$ is unitarily equivalent to the multiplier $|\xi|^\beta e^{i\,\mathrm{sgn}(\xi)\beta\pi/2}$, it is identically equivalent to the Mellin-based definition of the Hadamard fractional derivative $(x\frac{d}{dx})^\beta$ evaluated via the Balakrishnan spectral calculus. Thus, under this specific functional calculus, AAC provides a rigorous maximal realization of the Hadamard derivative.*

6 Beyond Lévy Semigroups: Oscillatory Kernels

To demonstrate that AAC is a fundamentally broader spectral framework than translation-invariant Lévy semigroups, we examine non-monotone kernels.

Let the memory kernel be purely oscillatory: $\phi(s) = \sin(\omega s)$ for $\omega > 0$. The associated multiplier is $m(\xi) = i\xi \int_0^\infty e^{-i\xi s} \sin(\omega s) ds$. Evaluated distributionally, the symbol is strictly real:

$$m(\xi) = \frac{\omega \xi}{\omega^2 - \xi^2}.$$

Because $m(\xi)$ is real-valued, $\mathrm{Re}(m(\xi))$ changes sign, meaning $-\mathcal{T}_\phi$ is *not* conditionally negative definite and cannot generate a contraction semigroup. However, $i\mathcal{T}_\phi$ is self-adjoint. By Stone's Theorem [2], $\mathcal{T}_\phi$ generates a strongly continuous *unitary group* $G(t) = \exp(it\mathcal{T}_\phi)$. Thus, the Cauchy problem $\partial_t f = i\mathcal{T}_\phi f$ yields a dispersive, Schrödinger-type unitary evolution on the scale domain. Consequently, AAC serves as a unified spectral classification framework encompassing both dissipative Markovian memory (Lévy dynamics) and dispersive, conservative memory.

7 Conclusion

Anchored Accumulation Calculus rigorously embeds multiplicative convolutions into the theory of continuous Fourier multipliers. By establishing closed maximal extensions via unitary maps, we provided a mathematically airtight derivation of the scale-invariant Lévy–Khintchine formula for completely monotone kernels. Furthermore, by formalizing the sectoriality of heavy-tailed kernels and the dispersive unitary groups generated by oscillatory kernels, AAC transcends heuristic fractional calculus, offering a complete spectral taxonomy of scale-covariant memory operators.

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